On Energy-Based Acoustic Source Localization for Sensor Networks

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Abstract—In this paper, energy-based localization methods for source localization in sensor networks are examined. The focus is on least-squares-based approaches due to a good tradeoff between performance and complexity. A suite of methods are developed and compared. First, two previously proposed methods (quadratic elimination and one step) are shown to yield the same location estimate for a source. Next, it is shown that, as the errors which perturb the least-squares equations are nonidentically distributed, it is more appropriate to consider weighted least-squares methods, which are observed to yield significant performance gains over the unweighted methods. Finally, a new weighted direct least-squares formulation is presented and shown to outperform the previous methods with much less computational complexity. Unlike the quadratic elimination method, the weighted direct least-squares method is amenable to a correction technique which incorporates the dependence of unknown parameters leading to further performance gains. For a sufficiently large number of samples, simulations show that the weighted direct solution with correction (WDC) can be more accurate with significantly less computational complexity than the maximum-likelihood estimator and approaches Cramér–Rao bound (CRB). Furthermore, it is shown that WDC attains CRB for the case of a white source.

Index Terms—Acoustic source localization, sensor networks.

I. INTRODUCTION

A common application of a sensor network framework is source localization. Source localization is inherent to many monitoring applications, such as those for wildlife or surveillance. We shall assume acoustic sources herein. Localization methods based on direction of arrival or time delay estimation have been previously developed [1]–[3]. These methods exploited the temporal information conveyed in multiple collected samples, requiring time series information from multiple sensors. The acquisition of the sampled signals entails the transmission of the raw time series (hence, we denote such schemes signal based) which can require significant wireless resources. A contrasting method is to use energy information, exploiting the fact that acoustic signal intensity attenuates with distance from the source. Least-squares solutions for energy-based methods can be found in [4] and [5] and a maximum-likelihood (ML) estimator was presented in [6].

In [7], a fast converging localization solution was proposed using projection-onto-convex-sets although the performance is inferior to the ML estimator in [6]. Energy estimates of the source are obtained at each sensor via averaging of the data samples; these single estimates are fused either in a centralized (transmitted to a fusion center) or decentralized fashion to form the final localization estimate. For some scenarios, signal-based methods may offer improved performance versus energy-based methods since the information conveyed in all samples are directly exploited (without averaging), but at the expense of larger transmission resources, e.g., wireless bandwidth. We also note that, for white sources, the energy observations are a set of sufficient statistics; however, for colored sources, they are not.

The scenario considered herein is depicted in Fig. 1, where all sensors measure the acoustic signal created by the acoustic source. The average energy is computed from the received signal before being transmitted (via a wireless channel represented by dashed arrow paths) to combine with the energy readings from other sensors at a fusion center where localization is conducted. The combining procedure can be carried out in a centralized or decentralized fashion.

Energy-based localization capitalizes on ideas similar to received signal strength (RSS)-based methods, that is, the signal energy decays with distance. However, the prior work on RSS-based methods has some significant differences with the work considered herein: the nodes themselves are to be localized [8] versus a source; the sensor nodes cooperate with each other, whereas a target source typically does not; thus, the transmitted energy is often assumed known; and...
the transmitted signals from the sensors can be designed to improve localization, whereas a source signal can have arbitrary characteristics. ML estimators based on RSS measurements can be found in [9] and [10], and a least-squares solution was presented in [11].

In the current work, we do not assume access to range information a priori. Unlike the signal models in [6] and [7], we consider an arbitrary acoustic source characterized by an unknown correlation function and, thus, derive more generalized statistical properties of energy observations. The source generates a signal experiencing attenuation modeled as in [6]. In contrast to the ML estimators of [6], which require iterative solutions and may demand high computational complexity (later discussed in Section VII), we focus on least-squares methods which offer a good tradeoff between performance and complexity. The least-squares approaches considered herein are limited to the case of single source, but their ease of implementation and feasibility for real-time scenarios motivates the investigation. The least-squares solutions for a signal model similar to that in [6] were previously reported in [5]. However, the solutions were based on the assumption that the errors that perturb the least-squares equations are i.i.d., zero-mean Gaussian random variables; herein we show this assumption to be inaccurate for our model. The main contribution of the work has two parts. First, we compare, contrast and improve some existing methods. We prove that the least-squares solution recently proposed in [5] based on energy ratio approach, referred to as the quadratic elimination (QE) algorithm, yields the same location estimate as the previously proposed one step (OS) least-squares solution [12]. We show that given our signal model, the errors that perturb the equations employed for the least-squares methods are not white, and, thus, weighted schemes should be employed. We note that the coloration of the noise occurs even when a white signal source is assumed. The need for a weighted schemes motivates the development of the weighted one step least-squares (WOS) method. Second, we introduce a new approach to formulate the least-squares problem which does not require the energy ratio computation and proposed weighted direct least-squares method (WD) that yields the same the location estimate as WOS; however, WD offers lower computational complexity. WOS and WD can employ a correction technique [13] which enables the incorporation of parameter dependencies leading to further performance gains, while the QE cannot make use of this correction method.

Simulation results demonstrate that, overall, WD and WOS significantly outperform OS and QE. WD performance is further improved when combined with a correction technique (WDC). As the number of samples employed increases, the simulations show that WDC approaches the Cramér–Rao bound (CRB) and can outperform ML estimators while requiring significantly less complexity. Furthermore, we prove that WDC attains the CRB for the case of a white source. The robustness of all schemes is investigated through the variation of system parameters including the consideration of errors in prior information. We note that the trends observed in the simulated acoustic source also hold true when the various methods are compared when using experimental source data based on real bird songs.

This paper is organized as follows. In Section II, the signal model is provided. The existing least-squares solutions for energy ratio-based methods are reviewed in Section III. The WOS least-squares solution is presented in Section IV. In Section V, the WD least-squares approach is developed and compared with WOS. The correction technique that can improve WOS and WD solutions is reviewed in Section VI. In Section VII, simulation results are provided with the aim of comparing the performance of different methods subject to significant parameter variations. Conclusions are given in Section VIII. Appendix I shows that QE and OS yield the same location estimate. WOS and WD are shown to yield the same location estimate in Appendix II. Appendix III derives the CRB for energy-based localization, and Appendix IV proves that WDC achieves the CRB when a large number of samples and a white source are assumed.

II. SIGNAL MODEL

A static acoustic source generating a wide sense stationary (WSS) Gaussian random process, $s(t) \sim N(0, \sigma_s^2)$, with the correlation function $R_s(t)$ is assumed. The intensity of the source attenuates at a rate that is inversely proportional to the distance from the source [14]. Given $N$ sensors, the received signal at the $i$th sensor is given by

$$x_i(t) = \frac{s(t - \tau_i)}{||r_i - r_i||^{1/2}} + w_i(t), \quad 1 \leq i \leq N$$

where $w_i(t)$ is white Gaussian measurement noise, $w_i(t) \sim N(0, \sigma_w^2)$. The vectors $r_i$ and $r_s$ denote the coordinates of the $i$th sensor and the source, respectively. In practice, $\sigma_w^2$ can be estimated based on measurements when the source is absent. For concise notation, $\sigma_w^2$ is assumed to be identical for all sensors, $\sigma_w^2 = \sigma_w^2$ for $1 \leq i \leq N$, and is assumed to be known. Note that for the case of different $\sigma_w^2$, the signal model is slightly changed and the considered algorithms require only simple modification. The time required for the acoustic signal to propagate from the source to the $i$th sensor is denoted by $\tau_i$. In [5], it was shown that the effective decay factor, $\alpha$, is approximately 2 and this value is assumed for this paper. In reality, however, $\alpha$ can deviate from the assumed value due to the environment. The sensitivity of the considered algorithms to this deviation is investigated via simulation in Section VII. Note that the model in (1) is limited to cases where reflections from the ground and reverberation effects are minimal, e.g., the large open space with porous ground such as grassland. The signal energy at the $i$th sensor can be calculated by averaging over a number of observations ($L$) sampled at frequency $f_s$

$$y_i = \frac{1}{L} \sum_{n=0}^{L-1} x_i^2 \left( t_s + \frac{n}{f_s} \right)$$

where $t_s$ is the starting time. The sensing process (collecting samples used for energy computation) starts when the presence of source is detected. Source detection schemes can be found in...
[15]. We assume that \( s(t) \) and \( u_i(t) \) are independent. Expanding (2) yields

\[
 y_i = \frac{1}{L} \sum_{n=0}^{L-1} \frac{s^2}{\| r_s - r_i \|^2} \left( t_s + \frac{n}{f_s} - \tau_i \right) + \frac{1}{L} \sum_{n=0}^{L-1} 2s \left( t_s + \frac{n}{f_s} - \tau_i \right) w_i \left( t_s + \frac{n}{f_s} \right) + \frac{1}{L} \sum_{n=0}^{L-1} w_i^2 \left( t_s + \frac{n}{f_s} \right).
\]

(3)

In [6], [7], and [16], the second term in (3) is assumed to be zero due to the independence between \( s(t) \) and \( u_i(t) \). However, if \( 1/L \| r_s - r_i \| \sum_{n=0}^{L-1} 2s(t_s + n/f_s - \tau_i)w_i(t_s + n/f_s) \approx 0 \), the other terms should also be deterministic, i.e., \( 1/L \sum_{n=0}^{L-1} w_i^2(t_s + n/f_s) \approx \sigma^2_w \) and \( y_i \) should not be modeled as a random variable. Herein, we consider all terms in (3). Thus, the first and second moments of \( y_i \) are given by

\[
 \mathbb{E}[y_i] \triangleq \mu_i = \frac{\sigma^2_s}{\| r_s - r_i \|^2} + \sigma^2_w \\
 \text{var}[y_i] \triangleq \sigma^2_i = \frac{1}{L} \left( 2 \sum_{n=1}^{L} \sum_{m=1}^{n-1} \frac{R_s^2 \left( \frac{(n-m)}{f_s} \right)}{\| r_s - r_i \|^2} + \frac{4L \sigma^2_s}{\| r_s - r_i \|^2} + 2L \sigma^2_w \right) \\
 \text{cov}[y_i, y_j] \triangleq \sigma_{ij} = \frac{2}{L} \sum_{n=1}^{L} \sum_{m=1}^{n-1} \frac{R_s^2 \left( \frac{(n-m)}{f_s} + \tau_j - \tau_i \right)}{\| r_s - r_i \|^2 \| r_s - r_j \|^2} \sum_{n=1}^{L} \sum_{m=1}^{n-1} \frac{R_s^2 \left( \frac{(n-m)}{f_s} \right)}{\| r_s - r_j \|^2}. 
\]

(4)

We make the assumption that the time delays, \( |\tau_j - \tau_i| \), across sensors are small compared to the observation time so that the approximation in the last step of (4) is obtained. We also assume that \( R_s(\delta) \to 0 \) as \( \delta \to \infty \) so that the Central Limit Theorem can be applied when \( L \to \infty \) [17]. Thus, \( y_i \) is approximately normal distributed.\(^2\) Note that, unlike the derivation in [6], [7], and [16], the derived variances and covariances not only are function of the noise variance, \( \sigma^2_w \), but also the correlation function for the source signal and the distance between sensors and the source. Therefore, the variance of the energy observations at each sensor can be varied significantly depending on the distances to the source. The energy-based approaches localize the source based on the observations, \( y_i \) for \( 1 \leq i \leq N \).

**III. LEAST-SQUARES SOLUTIONS FOR ENERGY RATIO APPROACH**

This section summarizes the least-squares solution based on the energy ratio approach recently presented in [5] and [16] and compares it with the previously proposed method [12]. The energy ratio between the \( i \)th and \( j \)th sensors is defined by

\[
 K_{ij} = \left( \frac{y_i - \sigma^2_s}{y_j - \sigma^2_s} \right)^{-1/2}. 
\]

(5)

In [5] and [16], the least-squares equation is formulated by initially assuming that \( y_i = \mu_i \); thus, we have the noiseless energy ratio as follows:

\[
 K_{ij} = \left( \frac{\mu_i - \sigma^2_s}{\mu_j - \sigma^2_s} \right)^{-1/2} = \left\| r_s - r_i \right\|. 
\]

(6)

Note that given \( N \) sensors, \( M \leq N(N-1)/2 \) pairs of energy ratios can be compute, i.e., when all possible pairs are used; we define the index and index set as follows, the index \( (i, j) \in I \) where the members of \( I \) are 2-subsets of \( \{1, \ldots, N\} \). According to (6), the source location, \( r_s \), must reside on the hypersphere described by the set of equations given by

\[
 \left\| r_s - c_{ij} \right\|^2 = \rho_{ij}^2, \quad \forall (i, j) \in I 
\]

(7)

where

\[
 c_{ij} = \frac{r_i - K_{ij} r_j}{1 - K_{ij}^2}, \quad \rho_{ij} = \frac{K_{ij} \left\| r_s - r_i \right\|}{1 - K_{ij}^2}. 
\]

(8)

An unconstrained least-squares solution is formulated by considering different pairs of hyperspheres; for example, a pair formed by \( K_{ij} \) and \( K_{kl} \)

\[
 \left\| r_s - c_{ij} \right\|^2 = \rho_{ij}^2, \quad \left\| r_s - c_{kl} \right\|^2 = \rho_{kl}^2. 
\]

(9)

Subtracting each side of these two equations to cancel the term \( \| r_s - c_{ij} \|^2 \) yields

\[
 \frac{2}{\alpha_i} (c_{ij}^T - c_{kl}^T) r_s = \frac{(c_{ij}^T - \rho_{ij}^2) - (c_{kl}^T - \rho_{kl}^2)}{b_i} 
\]

(10)

where \( 1 \leq n \leq P \), and \( P = M(M-1)/2 \) is the maximum number of pairs of equations (from a total of \( M \) equations formed by \( M \) energy ratios). In the presence of observation noise, the source location estimate can be obtained by minimizing the cost function as follows:

\[
 \hat{r}_s = \arg \min_{r_s} \left\{ J(r_s) = \sum_{i=1}^{P} \left\| c_{ij}^T r_s - b_i \right\|^2 \right\} = (G^T G)^{-1} G^T b 
\]

(11)

where \( G = [a_{i1}, a_{i2}, \ldots, a_{iP}]^T \) and \( b = [b_1, b_2, \ldots, b_P]^T \). If \( N \geq 3 \), \( P \geq 2 \), \( G \) is full column rank, and a unique \( \hat{r}_s \) can be obtained. However, when noise is present, \( G \) and \( b \) are computed from \( \hat{K}_{ij} \) instead of the noiseless \( K_{ij} \) which is not available. Thus, there is a possibility of ill-conditioned cases, e.g., \( G \) is not full column rank. Such cases are very unlikely particularly when \( N > 3 \). We assume \( N \geq 3 \) and the cases are not ill-conditioned through out the paper. The solution in (11)

\(^2\)Numerical results show that when the number of samples is larger than 2500 and the signal has significant spectral content up to \( f_s/4 \), the modified Lilliefors normality test (using Matlab function [18]) does not reject the null hypothesis for the significance level = 0.01, i.e., our approximation is good.

\(^3\)The ill-conditioned cases never happened in our simulations where \( N > 6 \).
was denoted the QE in [4] and we label the location estimate obtained via QE by \( \hat{r}_{s,\text{QE}} \).

**A. QE Versus One Step Least-Squares Solution**

The QE solution is now compared with another least-squares solution [12], denoted the OS solution, which was proposed for range difference-based localization. The solution in [12] can be adapted to the energy ratio approach as follows. Expanding a hypersphere equation, \( \| r_s - c_{ij} \|^2 = \rho_{ij}^2 \) gives

\[
\begin{align*}
\| r_s \|^2 - 2c_{ij}^T r_s + \| c_{ij} \|^2 &= \rho_{ij}^2 \\
\left[ 2c_{ij}^T - 1 \right] \frac{r_s}{\| r_s \|^2} &= \left( c_{ij} \right)^T - \rho_{ij}^2 \notag
\end{align*}
\]

where \( 1 \leq m \leq M \). A set of equations can be formed and written in matrix form as follows:

\[
G \theta_{OS} = b
\]

(13)

where \( G = [d_1, d_2, \ldots, d_M]^T \), \( \theta_{OS} = [r_{s,1}, \| r_s \|^2]^T \) and \( b = [b_1, b_2, \ldots, b_M]^T \). The parameter estimate, \( \theta_{OS} \), is given by

\[
\hat{\theta}_{OS} = (G^T G)^{-1} G^T b
\]

(14)

and the location estimate using OS is denoted by \( \hat{r}_{s,\text{OS}} \). Note that under non-ill-conditioned cases, \( G \) is of rank 3, and, thus, is full column rank. The relationship between QE and OS can be seen through the conversion of equations used by both method, i.e., each equation used by QE can be constructed from a pair of equations used by OS. Note that the number of equations used for OS is \( M = N(N - 1)/2 \) while it is \( P = M(M - 1)/2 \) for QE where \( N \) is the number of sensors. Appendix I shows that QE and OS yield the same location estimate.\(^4\) However, since QE does not estimate \( \| r_s \|^2 \), the correction technique [13] (explained in Section VI) that incorporates the dependence between \( r_s \) and \( \| r_s \|^2 \) in order to improve the localization performance can not be applied, and, thus, QE can be less accurate than a corrected OS. Hence, we will focus on OS and show that its performance can be improved further.

**IV. WEIGHTED ONE STEP LEAST-SQUARES SOLUTION**

In this section, we derive an optimal weighting for a weighted least-squares solution to improve the originally proposed OS method [12] since we found that, given our signal model in (1), the equation noise (later called the **LS equation error**) perturbing (13) is not white. Squaring both sides of (5) gives

\[
\hat{K}_{ij} = \frac{y_j - \sigma_{\text{w}}}{y_k - \sigma_{\text{w}}} = \mu_{ij} - \sigma_{\text{w}}^2 + e_{ij} = \left( \frac{\| r_s - r_j \|^2}{\| r_s - r_j \|^2} + e_{ij} \right)
\]

where \( e_{ij} \) is the error due to the deviation of \( y_k \) and \( y_j \) from \( \mu_i \) and \( \mu_j \). Equation (15) can be rearranged as follows:

\[
\frac{2\hat{K}_{ij} r_s - \| r_s \|^2}{\| r_s \|^2 - \frac{1}{2} \hat{K}_{ij} r_s} = \left( \frac{\| r_s - r_j \|^2}{\| r_s - r_j \|^2} + \frac{e_{ij}}{e_{ij}} \right) \cdot \frac{\| r_s - r_j \|^2}{1 - \hat{K}_{ij}^2}
\]

(16)

(16) is the same as (12), but the error that perturbs the equation (\( e_{ij} \)) when formed by the energy ratios computed from observations, is explicitly shown. This error term is referred to as the **LS equation error**. We make the following observations on \( e_{ij} \). First, the variance of \( e_{ij} \) is dependent on the distance between sensors and the source, \( \| r_s - r_j \|^2 \). Second, energy ratios computed from common sensors, e.g., \( \hat{K}_{ij} \) and \( \hat{K}_{kj} \), are correlated and that causes the \( e_{ij} \) and \( e_{kj} \) to be correlated. Thus, we consider a weighted least-squares solution [19]. The weighting applied to improve OS should be \( C_{\text{w}}^{-1} \) where \( C_{\text{w}} \) is the covariance matrix for \( \epsilon = [\epsilon_{(1)}(t), \epsilon_{(2)}(t), \ldots, \epsilon_{(M)}(t)]^T \) [19] and the subscript \((m)\), \( 1 \leq m \leq M \), sequentially represents \( \epsilon \) that appears in the equation formed by the different energy ratios, i.e., \( \epsilon_{(m)}(t) = \epsilon_{m,12} \) if (16) is the first equation in the set of equations. \( C_{\text{w}} \) is determined as follows. Let \( \epsilon_{(1)} = \epsilon_{m,12} \) and \( \epsilon_{(v)} = \epsilon_{m,12} \) be the LS equation errors that perturb the equation formed by the energy ratio \( \hat{K}_{m,12} = y_{12} - \sigma_{\text{w}}^2 / y_{12} - \sigma_{\text{w}}^2 \) and \( \hat{K}_{m,12} = y_{12} - \sigma_{\text{w}}^2 / y_{12} - \sigma_{\text{w}}^2 \) respectively. For convenience, \( \| r_s - r_j \|^2 \) is denoted as \( d_i \). Inserting \( \epsilon_{m,12} = \hat{K}_{m,12} - d_i / d_{12} \) and \( \epsilon_{m,12} = \hat{K}_{m,12} - d_i / d_{12} \) [see (15)] into the expressions for \( \epsilon_{m,12} \) and \( \epsilon_{m,12} \) [see (16)] gives

\[
\epsilon_{m,12} = d_i y_{12} - d_i y_{12} + d_i y_{12} - y_{12} = \frac{d_i y_{12} - y_{12}}{y_{12} - y_{12}}
\]

(17)

\[
\epsilon_{m,12} = d_i y_{12} - d_i y_{12} + d_i y_{12} - y_{12} = \frac{d_i y_{12} - y_{12}}{y_{12} - y_{12}}
\]

(18)

Expanding \( \epsilon_{m,12} \) and \( \epsilon_{m,12} \) into a Taylor series about \( \{ \mu_{12}, \mu_{12} \} \) and \( \{ \mu_{12}, \mu_{12} \} \), and using the first term to approximate the mean value yields

\[
\mathbf{E}\{\epsilon_{m,12}\} \approx \mathbf{E}\{\epsilon_{m,12}\} \approx 0
\]

(19)

By neglecting moments of orders higher than 2, the covariance
of \( \epsilon_{\text{w1} v_1} \) and \( \epsilon_{\text{w1} v_2} \) can be approximated as follows:

\[
[C_e]_{\text{w1}} \simeq \text{cov}\{y_{\text{w1}}, y_{\text{w2}}\} \left( \frac{\partial y_{\text{w1}}}{\partial y_{\text{w1}}} \right) \left( \frac{\partial y_{\text{w2}}}{\partial y_{\text{w2}}} \right) + \text{cov}\{y_{\text{w1}}, y_{\text{v2}}\} \left( \frac{\partial y_{\text{w1}}}{\partial y_{\text{v1}}} \right) \left( \frac{\partial y_{\text{v2}}}{\partial y_{\text{v1}}} \right) + \text{cov}\{y_{\text{v1}}, y_{\text{w2}}\} \left( \frac{\partial y_{\text{v1}}}{\partial y_{\text{w1}}} \right) \left( \frac{\partial y_{\text{v2}}}{\partial y_{\text{w2}}} \right) + \text{cov}\{y_{\text{v1}}, y_{\text{v2}}\} \left( \frac{\partial y_{\text{v1}}}{\partial y_{\text{v1}}} \right) \left( \frac{\partial y_{\text{v2}}}{\partial y_{\text{v2}}} \right)
\]

where the derivatives are evaluated at \( \{y_{\text{w1}}, y_{\text{w2}}, y_{\text{v1}}, y_{\text{v2}}\} \). Even though \( [C_e]_{\text{w1}} \) are functions of \( \{y_{\text{w1}}, y_{\text{w2}}, y_{\text{v1}}, y_{\text{v2}}\} \) which are dependent on \( r_s \), and thus, are not available in practice, they are approximated by \( \{y_{\text{w1}}, y_{\text{w2}}, y_{\text{v1}}, y_{\text{v2}}\} \) to obtain an estimate of \( C_e \). The common scaling of the elements of the weighting can be neglected without affecting the location estimate. Thus, the weighting becomes \( C_e^{-1} \) where \( C_e \) is defined as shown in (21), at the bottom of the page. The WOS solution, thus, is given by

\[
\hat{\theta}_{\text{WOS}} = \left( \hat{G}^T C_e^{-1} \hat{G} \right)^{-1} \hat{G}^T C_e^{-1} \hat{h}.
\]

It should be pointed out here that \( \hat{C}_e \) is shown to be singular (see Appendix II) and the pseudoinverse of \( \hat{C}_e \), \( \hat{C}_e^\dagger \), is used instead of \( \hat{C}_e^{-1} \). Note that the singularity occurs due to the approximation in (20); thus, it does not truly represent the characteristics of \( C_e \). The WOS is based on an energy ratio approach and will be shown to be equivalent to a simple, direct localization method that does not require the energy ratio computations in the next section.

V. WEIGHTED DIRECT LEAST-SQUARES SOLUTION

The idea of the direct approach is to localize the source directly based on the measured energy \( [y_i] \) in (2)) without computing the energy ratios [see (5)]. The observation \( y_i \) can be written as

\[
y_i = \frac{\sigma^2_i}{||r_s - r_i||^2} + \sigma^2_w + \epsilon_i
\]

where \( \epsilon_i \) is the LS equation error and is approximately normal distributed as explained in Section II where its first and second moments are shown in (4). Equation (23) can be rearranged as follows:

\[
2 \left( \sigma^2_w - y_i \right) r_i^T r_s + \left( y_i - \sigma^2_w \right) ||r_s||^2 - \sigma^2_w = \left( \sigma^2_w - y_i \right) ||r_i||^2 + \epsilon_i ||r_s - r_i||^2.
\]

Considering the \( N \) sensors together and writing these equations in matrix form yields

\[
A \theta_{\text{WD}} = h + \epsilon
\]

where

\[
A = \begin{bmatrix}
-2 (y_i - \sigma^2_w) r_i^T & (y_i - \sigma^2_w) \quad -1 \\
\vdots & \vdots \\
-2 (y_N - \sigma^2_w) r_N^T & (y_N - \sigma^2_w) \quad -1 
\end{bmatrix}
\]

\[
\theta_{\text{WD}} = \begin{bmatrix}
||r_s - r_1||^2, \ldots, ||r_s - r_N||^2
\end{bmatrix}^T
\]

\[
h = \begin{bmatrix}
-||r_i - \sigma^2_w||^2, \ldots, -||r_N - \sigma^2_w||^2
\end{bmatrix}^T
\]

\[
\epsilon = \begin{bmatrix}
\epsilon_1 ||r_s - r_1||^2, \ldots, \epsilon_N ||r_s - r_N||^2
\end{bmatrix}^T.
\]

Note that \( \sigma^2_w \) [the variance of the source signal (see Section II)] is also an unknown parameter which, for the energy ratio approach, is canceled during the energy ratio computation. The estimate can be obtained using the weighted least-squares solution (the WD solution), where the covariance matrix for \( \epsilon \) needs to be determined. According to (4) and (23), the covariance matrix for \( \epsilon \) is given by

\[
C_e = \frac{2}{L^2} \left( Q + \beta 1_N 1_N^T \right)
\]

where

\[
Q = 2 L \sigma^2_w \text{diag} \left( \frac{2 \sigma^2_w}{||r_s - r_1||^2} + \sigma^2_w \ldots \right)
\]

\[
\beta = \sum_{n=1}^{L} \sum_{m=1}^{L} \left( \frac{n - m}{f_s} \right)^2 \left( \frac{1}{N} \right)^T.
\]
Using the Woodbury matrix identity [20] to expand $C^{-1}$, neglecting the scaling, and approximating $Q$ using $\mu_i \simeq y_i$, $Q = \text{diag}(2y_i - \sigma_{2i}^2/y_i - \sigma_{2i}^2), \ldots, 2y_N - \sigma_{2i}^2/y_N - \sigma_{2i}^2)$, yields the WD solution

$$\hat{\theta}_{\text{WD}} = \left( A^T Q^{-1} A \right)^{-1} A^T Q^{-1} \mathbf{h}. \quad (27)$$

Similar to the comparison between QE and OS presented in Appendix I, the relationship between WOS and WD can be analyzed based on the conversion between the set of equations used by both methods. Appendix II shows that WD and WOS, in fact, yield the same location estimate. Regarding considerations on complexity, $O(N^5)$ operations are required to construct the singular value decomposition (SVD) in order to obtain the weighting for WOS [20]. The number of multiplications and additions to multiply the matrix for WOS is $O(N^4)$. Hence, the computational complexity demanded for WOS is $O(N^5)$. For WD, the weighting is diagonal with dimension $N \times N$, thus, to obtain the weighting requires only $O(N)$ operations. The multiplications and additions used for WD are $O(N^2)$. In total, the computational complexity required for WD is $O(N^2)$. Therefore, although yielding the same location estimate, WD is more computationally efficient than WOS; and WD should be applied instead of WOS in all scenarios.

WD can be implemented in a distributed fashion by developing a sequential least-squares solution similar to what was proposed in [21] wherein such a solution was used in the context of time delay-based localization. The sequential solution allows the information of sensor locations and energy observations to be kept at each sensor and used to update estimated source location which is passed through the selected route.

VI. Correction Technique

Formulation of $\hat{\theta}_{\text{WOS}}$ and $\hat{\theta}_{\text{WD}}$ is made under the assumption that $||r_s||^2$ is a scalar that is statistically independent of $r_s$; clearly this is not true. The relationship can be incorporated in the formulation and used to improve the estimate derived by the approach developed in [13]. The steps for applying the correction technique to WOS and WD are similar. Since both methods yield the same location estimate where WD is more computationally efficient, the required steps for WD will be explained and the corrected solution is called WDC. Let $\hat{\theta}_{\text{WD}}$ be expressed as $\hat{\theta}_{\text{WD}} = \hat{\theta}_{\text{WD}} + \Delta \theta$. When $y_i$ is of low variance [having sufficiently large number of observation samples, $L$, see (4)], $Q$ (see (27)) is close to $Q$ in (26) without the scaling $2L\sigma_{2i}^2\sigma_{2i}^2$; the bias and covariance matrix of $\hat{\theta}_{\text{WD}}$ can be approximated by [13]

$$\Delta \theta \simeq \left( A^T C_{\varepsilon}^{-1} A \right)^{-1} A^T C_{\varepsilon}^{-1} \varepsilon, \quad C_{\Delta \theta} \simeq \left( A^T C_{\varepsilon}^{-1} A \right)^{-1} \quad (28)$$

where $A$ represents $E[A]$, i.e., $y_i$ in $A$ is replaced by $\mu_i$. Since $\varepsilon$ has zero mean, $E[\Delta \theta] = 0$ and $\hat{\theta}_{\text{WD}}$ can be considered as a random vector with its mean centered at the true value and the covariance matrix given by (28). Thus, the elements of $\hat{\theta}_{\text{WD}}$ can be expressed as

$$\hat{\theta}_{\text{WD},1} = x_s + c_1, \quad \hat{\theta}_{\text{WD},2} = y_s + c_2, \quad \hat{\theta}_{\text{WD},3} = ||r_s||^2 + c_3, \quad \hat{\theta}_{\text{WD},4} = \sigma_{2s}^2 + c_4 \quad (29)$$

where $c_1, c_2, c_3$, and $c_4$ are estimation errors of $\hat{\theta}_{\text{WD}}$ and $r_s = (x_s, y_s)$ is the true source location. Squaring $\hat{\theta}_{\text{WD},1}$ and $\hat{\theta}_{\text{WD},2}$ gives

$$\hat{\theta}_{\text{WD},1}^2 = x_s^2 + 2x_s c_1 + c_1^2 \simeq x_s^2 + 2x_s c_1 \quad (30)$$

$$\hat{\theta}_{\text{WD},2}^2 = y_s^2 + 2y_s c_2 + c_2^2 \simeq y_s^2 + 2y_s c_2 \quad (31)$$

where the approximation is valid when $c_1$ and $c_2$ are small. A set of equations $A\hat{\theta}^2 = \hat{\mathbf{h}} + \hat{\mathbf{e}}$ is obtained where

$$\hat{\mathbf{A}} = \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{bmatrix}, \quad \hat{\theta}^2 = \begin{bmatrix} x_s^2 & y_s^2 & \sigma_{2s}^2 \end{bmatrix}^T$$

$$\hat{\mathbf{h}} = \begin{bmatrix} \hat{\theta}_{\text{WD},1}^2 \\
\hat{\theta}_{\text{WD},2}^2 \\
\hat{\theta}_{\text{WD},3} \\
\hat{\theta}_{\text{WD},4} \end{bmatrix}^T$$

$$\hat{\mathbf{e}} = \begin{bmatrix} 2x_s c_1 \\
2y_s c_2 \\
c_3 \\
c_4 \end{bmatrix}^T \quad (32)$$

Since $\Delta \theta = [c_1 \ c_2 \ c_3 \ c_4]^T$ (see (29)) and $\hat{\mathbf{e}} = B\Delta \theta$ where $B = \text{diag}(2x_s, 2y_s, 1, 1)$ [see (32)], $\hat{\mathbf{e}} \sim N(0, \Psi)$ and $\Psi = B\Sigma\sigma B$. Therefore, the ML estimate for $A\hat{\theta}^2 = \hat{\mathbf{h}} + \hat{\mathbf{e}}$ is given by

$$\hat{\theta}^2 \equiv \begin{bmatrix} x_s^2 & y_s^2 & \sigma_{2s}^2 \end{bmatrix}^T$$

$$= \left( \hat{\mathbf{A}}^T \Sigma^{-1} \hat{\mathbf{A}} \right)^{-1} \hat{\mathbf{A}}^T \Sigma^{-1} \hat{\mathbf{h}}$$

$$= \left( \hat{\mathbf{A}}^T B^{-1} \hat{\mathbf{A}}^T C_{\varepsilon}^{-1} A \hat{\mathbf{B}} - \hat{\mathbf{A}} \right)^{-1} \hat{\mathbf{A}}^T B^{-1} \hat{\mathbf{A}}^T C_{\varepsilon}^{-1} A \hat{\mathbf{B}} - \hat{\mathbf{A}} \hat{\mathbf{h}}$$

$$= \left( \hat{\mathbf{A}}^T B^{-1} \hat{\mathbf{A}} \right)^{-1} \hat{\mathbf{A}}^T \left( Q + \beta_1 N \tilde{\mathbf{X}}_s \right)^{-1} \hat{\mathbf{A}}^T \hat{\mathbf{B}}^T \hat{\mathbf{A}} \hat{\mathbf{h}}$$

$$= \left( \hat{\mathbf{A}}^T B^{-1} \hat{\mathbf{A}} \right)^{-1} \hat{\mathbf{A}}^T \hat{\mathbf{Q}}^{-1} \hat{\mathbf{A}}^T B^{-1} \hat{\mathbf{A}} \hat{\mathbf{h}}$$

$$= \left( \hat{\mathbf{A}}^T B^{-1} \hat{\mathbf{Q}}^{-1} \hat{\mathbf{A}} \right)^{-1} \hat{\mathbf{A}}^T B^{-1} \hat{\mathbf{A}} \hat{\mathbf{h}} \quad (33)$$

The second to last step in (33) is obtained by using Woodbury matrix identity and the approximation in the last step uses the results of WD to estimate the elements in $B$ to get $\hat{B}$, and $\hat{\mathbf{A}}$ and $\hat{\mathbf{Q}}$ were defined in Section V. The final location estimate incorporated with the parameter dependence is $\hat{\theta}_{\text{WDC}} = \left[ \pm \sqrt{x_s^2} \pm \sqrt{y_s^2} \right]$. The proper solution is selected to be the one which lies in the region of interest, i.e., closest to $\hat{\theta}_{\text{WD}}$. The performance of WDC compared with other schemes including the CRB will be shown via simulations in Section VII. However, for the case of white source and large number of samples, we show that the WDC attains the CRB in Appendix IV. As pointed out earlier, the previously proposed QE method, although yielding the same location estimate as OS, can not be improved by the correction technique since the unknown parameter, $||r_s||^2$, is eliminated. Consequently, the corrected OS can be more accurate than QE when the correction technique is appropriately applied, e.g., when the number of samples is large.
VII. SIMULATION RESULTS

The performance of the OS, WOS, WD, WDC, and ML methods compared to the CRB (see Appendix III) was investigated through a series of Monte Carlo simulations. A number of sensors, \( N \), were placed randomly and uniformly in the region of interest sized 25 \( \times \) 25 square meters. The location of a static acoustic source was assumed to have a uniform distribution within the sensor field. 100 different sensor topologies with 1000 trials per topology were used to average the noise and the effect of sensor locations. The quality of the estimators were evaluated by computing the root mean square error (RMSE) defined as follows:

\[
\text{RMSE} = \sqrt{\frac{\sum_{i=1}^{N_t} \sum_{j=1}^{N_s} |\hat{\mathbf{r}}_{i,j} - \mathbf{r}_i|^2}{N_t N_s}} \tag{34}
\]

where the index \( i \) and \( j \) correspond to the results from the \( i \)th trial and the \( j \)th topology, \( N_t = 100 \), and \( N_s = 1000 \). The CRB was computed as follows:

\[
\text{CRB} = \sqrt{\frac{\sum_{i=1}^{N_t} [\mathbf{F}^{-1}]_{11} + [\mathbf{F}^{-1}]_{22}}{N_t}} \tag{35}
\]

where \( \mathbf{F}^{-1} \) is derived in Appendix III. For OS and WOS, all possible \( N(N-1)/2 \) energy ratios were used for the localization. The ML estimate was obtained by maximizing the following likelihood function:

\[
\max_{\hat{\boldsymbol{\theta}}} \left\{ \mathcal{L}(\hat{\boldsymbol{\theta}}) \right\} = \max_{\hat{\boldsymbol{\theta}}} \left\{ \ln |\text{det} \mathbf{C}(\hat{\boldsymbol{\theta}})| \right. \\
+ (\mathbf{y} - \mu(\hat{\boldsymbol{\theta}}))^T \mathbf{C}^{-1}(\hat{\boldsymbol{\theta}}) (\mathbf{y} - \mu(\hat{\boldsymbol{\theta}})) \right\} \tag{36}
\]

where \( \hat{\boldsymbol{\theta}} = [x_0, y_0, \sigma^2, \nu]^T \), \( \nu \) is defined in (62), Appendix III, and \( \mathbf{C} \) is the covariance matrix for energy measurements, \( \mathbf{y} = [y_1, \ldots, y_N]^T \), whose means are \([\mu_1, \ldots, \mu_N]^T\) (see Section II). This nonlinear optimization was iteratively solved until convergence using the Nelder–Mead nonlinear minimization [22] Matlab function where each iteration requires \( O(N^3) \) operations. Note that this ML estimator is somewhat different from the one proposed in [6] due to the different assumptions in the signal model as explained in Section II [between (3) and (4)]. In [6], the unknown parameters are \([x_0, y_0, \sigma^2, \nu]^T\) and the covariance matrix is a function of only the noise variance, \( \sigma^2 \). Herein, there is an additional unknown parameter, \( \nu \), which is dependent on the autocorrelation of the source. Moreover, the covariance matrix, \( \mathbf{C}(\hat{\boldsymbol{\theta}}) \), is a function of not only \( \sigma^2 \), but also unknown parameters, \( \sigma^2 \) and \( \nu \). This likelihood function is nonconvex, and, thus, the optimization may reach a local optima if initial estimates are far away from the true values. We used two different grid resolutions, \( 1 \times 1 \) and \( 5 \times 5 \) square meters labeled as ML-1 and ML-5, respectively, to perform the exhaustive search for the initial estimates. We observe that the resolution of the grid search will impact the quality of the estimates. \( \sigma^2/\sigma^2_0 \), defined as SNR\(_0\), was fixed to be 20 dB and the number of samples used to compute the energy \( (L) \) was 5000. Note that the Signal to Noise Ratio (SNR) at each sensor attenuates from SNR\(_0\) due to the distance (meters) between the source and the sensors according to the signal model. In other words, SNR\(_0\) is SNR measured at a distance of 1 meter from the source. The sampling frequency was set to be 5000 Hz and the bandwidth of the source was 0–1250 Hz.

The performance of each method when different numbers of sensors were deployed, \( 6 \leq N \leq 10 \), is illustrated in Fig. 2. The results show that WOS and WD perform equivalently and significantly outperform OS and QE. The improvement of WOS and WD over QE and OS becomes larger when the number of sensors is increased. This is because the variance and covariance of the energy ratios can be very large when the means of the received energy at sensor pairs are close [see (21) for the approximated expressions] and the possibility of this occurrence is higher when the number of sensors gets larger. Thus, some energy ratios may produce very inaccurate observations and significantly degrade the performance of OS and QE that equally rely on each energy ratio. The results from WD are improved when the correction technique is applied (WDC). ML-1 performs better than WDC but the difference becomes smaller when the number of sensors is increased. ML-5 becomes worse than other schemes except for QE and OS with number of sensors more than 7. This is because the performance of ML-5 is limited by the resolution of the exhaustive.

Other than the number of sensors, the performance with the variation of other significant system parameters was explored. Since the weightings for both WOS and WD are computed from the observations, the errors of the energy readings might lead to inaccurate weightings that can cause degradation instead of improvement. Also, WDC can be affected by the inaccurate estimate obtained from WD. According to the signal model, SNR\(_0\) and \( L \) are the parameters that influence the accuracy of the measured energy. Experiments were set up to investigate the robustness of each method with respect to variation in these parameters. SNR\(_0\) was set to be varied from 5, 10, \ldots, 30 dB where \( L = 5000 \) and \( N = 10 \). The results in Fig. 3 illustrate that...
even though WDC performs best when SNR$_0$ is high, it becomes worse than WD and WOS when SNR$_0$ is less than 10 dB. Therefore, the bad energy readings due to low SNR$_0$ can deteriorate the correction. The performance of WD and WOS is better than QE and OS for the entire range, but is degraded with a higher rate and finally converges to that of QE and OS with low SNR$_0$ (5 dB). ML-1 outperforms other schemes while ML-5 gains the least improvement over increasing SNR$_0$.

The number of samples used to compute the energy ($L$) is important for the system design since requiring fewer samples can save sensing power. However, a small $L$ increases the variance of the observed energy as described in (4). Fig. 4 illustrates the localization performance when $L$ is varied. SNR$_0$ is fixed to be 20 dB, and $N = 10$. WOS, WD, and WDC are shown to be better than OS for a wide range of $L$. However, when $L$ is smaller than 10, OS becomes superior to weighted schemes. This demonstrates that the weightings for WOS and WD are significantly affected by small $L$ and they are not appropriate for the case of very small number of samples. However, the case of using less than ten samples for energy computation in this scenario is not practical since it produces too large an error (RMSE > 10 m for $25 \times 25$-m field). When $L$ is very large, WDC approaches the CRB and is more accurate than ML-1 since ML-1, for some trials, reaches the local maxima due to the nonconvex likelihood function. Considering the complexity when $L = 10000$, ML-1 requires $26 \times 26 = 676$ trials ($O(N^3)$ operations/trial) for the initial estimate and uses an average of 54 iterations ($O(N^3)$ operations/iteration) to converge while WDC uses only $2 \times O(N^2)$ for the steps of WD and the correction. Thus, WDC is preferable to ML estimators under this condition. Tables I–III report the biases of all schemes with different parameters. The biases shown in the tables are $\sum_{i=1}^{N_s} \sum_{j=1}^{N_r} \left( \bar{r}_{s,i} - \bar{r}_s \right)^T / N_rN_s$. It is shown that the biases for all estimators are fairly small particularly with large number of sensors and samples, and ML estimators are less biased than the least-squares methods. We also found that if the source is located in outside the area that the sensors are located, the sensors (using all schemes) will bias the solution toward the sensor area.

The error of prior information can also affect the robustness of the localization. The deviation of the true values of noise variance ($\sigma_{\text{noise}}^2$), sensor locations, and decay factor ($\alpha$) from the assumed values is the parameter of interest for this experiment. The deviation is assumed to be uniformly distributed within the interval $2\delta$ centered at the assumed values. $\delta$ is set to be $0, 0.2, 0.3, 0.4, 0.5$ for noise variance, $0, 0.2, 0.3, 0.4, 0.5$ for decay factor, and $0, 1, 2, 3, 4, 5$ for sensor locations. The results in Figs. 5–7 show that WOS and WD are still better than QE and OS with the parameter deviation where WDC is degraded to be inferior to WD when the deviation gets larger. ML suffers less than other...
schemes under the deviations of noise variance and sensor locations but is degraded more than WOS and WD with the deviation of decay factor.

The aforementioned simulation results were obtained when an artificially generated source signal is used (predetermined PSD). Next, we investigate the performance of each scheme when a real acoustic source, a prerecorded bird song (Common Loon bird in Jasper National Park, Canada [23]), is used. The characteristics of this source deviate from the assumptions previously made, e.g., it may not be zero-mean WSS Gaussian random process. The sampling frequency for this recording is 11 KHz. The power spectral density of the bird song is shown in Fig. 8. The number of sensors is set to be 10 with uniformly random locations over 1000 trials, number of samples is 5000, and SNR is 20 dB. For each trial, the interval of collected samples is randomly selected from the total 109,052 samples of the entire record. The RMS error produced by each method is presented in Table IV. The results show that despite the deviation from the assumptions, WOS and WD still significantly outperform QE and OS. Although ML estimators perform better, ML-5 and ML-1 require 73 and 56 iterations on average ($O(N^2)$ operations/iteration), respectively. This is very high complexity compared with $2 \times O(N^2)$ operations demanded by WDC.

VIII. CONCLUSION

In this paper, a set of source localization methods based on energy measurements are developed, compared and contrasted. Two previously proposed methods, the QE least-squares solution for the energy ratio approach and the OS least-squares solution are shown to yield the same location estimate. In the pres-
ence of additive Gaussian noise, it is shown that OS can be improved by the WOS approach, which perturbs the least-squares equations so that the errors that perturb the least-squares equations are not white. We introduce a new least-squares method: the WD approach; WD is shown to yield the same location estimate as WOS with much lower computational complexity. The simulation results illustrate that WD and WOS significantly outperform QE and OS in most scenarios. WD can be further improved when combined with WDC. As the number of samples employed increases, WDC approaches the CRB and can be more accurate than ML estimators, but requires less computation. We also proved that WDC attains the CRB for the case of white source.

**APPENDIX I**

This Appendix shows that QE and OS yield the same location estimate. The set of equations used for QE (10) can be obtained from those used for OS (12). For example, for a given $M$ number of equations used for OS, multiplying a row vector $[1 -M 0 \ldots 0]$ both sides of (13) gives one equation used for QE. As many as $M(M-1)/2$ equations can be constructed from $M$ equations used by OS by multiplying both sides of (13) by the matrix $P$ whose rows contain 2 nonzero elements, 1 and -1, and 0 otherwise. The columns of the nonzero elements of each row are different resulting in $M(M-1)/2$ distinct rows.

Let $\hat{G}$ in (13) be $\begin{bmatrix} C & -1_M \end{bmatrix}$ where $1_M = \left[\frac{1, \ldots, 1}{M}\right]$ [see (12)]. Multiplying both sides of (13) by $P$ gives

\[
\begin{bmatrix}
PC0 \\
||r_s||^2
\end{bmatrix} = Pb
\]

\[
P^T C r_s = Pb.
\]

Hence, the least-squares solution for QE method is

\[
\hat{r}_{s,QE} = \left((PC)^T PC\right)^{-1}(PC)^T Pb.
\]

(39)

Note that the above solution is in fact (11) where $G$ is now shown as a function of $P$ and $C$ which is a block matrix in $\hat{G}$ and $C$ is full column rank $M$ by 2 matrix.

The OS solution given in (14) can be written in partitioned matrix as follows:

\[
\hat{\theta}_{OS} = \begin{bmatrix}
C^T C & -C^T 1_M \\
-1_M C & 1_M^T 1_M
\end{bmatrix}^{-1}
\begin{bmatrix}
C^T \\
-1_M^T
\end{bmatrix} b
\]

\[
= \begin{bmatrix}
Q_{11} & Q_{12} \\
Q_{21} & Q_{22}
\end{bmatrix}
\begin{bmatrix}
-1_M^T \\
1_M^T
\end{bmatrix} b
\]

\[
= \begin{bmatrix}
Q_{11}C^T & -Q_{12} 1_M^T \\
Q_{21}C^T & -Q_{22} 1_M^T
\end{bmatrix} b
\]

where

\[
Q_{11} = \left(C^T \left(I - \left(\frac{1}{M}\right) 1_M 1_M^T\right) C\right)^{-1}
\]

\[
Q_{12} = Q_{11} C^T 1_M \left(\frac{1}{M}\right).
\]

Although $\hat{\theta}_{OS}$ gives the estimate for both $r_s$ and $||r_s||^2$, to compare with $\hat{r}_{s,QE}$, only the location estimation, $\hat{r}_s$, is considered as follows:

\[
\hat{r}_{s,OS} = \left(Q_{11} C^T - Q_{12} 1_M^T\right) b.
\]

(40)

Substituting $Q_{11}$ and $Q_{12}$ and using $P^T P = MI - 1_M 1_M^T$ gives

\[
\hat{r}_{s,OS} = (C^T P^T P C)^{-1} C^T P^T Pb = \hat{r}_{s,QE}
\]

and, comparing (39) and (41), we see that QE and OS yield the same location estimate.

**APPENDIX II**

This Appendix shows that WOS and WD yield the same location estimate. Each equation used for WOS (16) is obtained from those used for WD (24) through the following conversion:

\[
\hat{p}_{ij} A \hat{\theta}_{WD} = \hat{p}_{ij}(h + e)
\]

(42)

where $\hat{p}_{ij} = [-1, 0, -1, 0, \ldots, 0, 0]$ and $\hat{p}_{ij} = [0, 0, 0, 0, \ldots, 0, 0]$. Note that if some energy observations have the same quantities, i.e., $y_i = y_j$, we can not pair these observations to form equations for WOS. We assume that this situation does not occur and it, in fact, never occurred in the considered simulation. Stacking up all possible $\hat{p}_{ij}$’s that construct $N(N-1)/2$ equations for WOS gives a conversion matrix $\hat{P}$ which is of rank $N-1$. Letting $A = [B -1_N]$ where $1_N = [1, \ldots, 1]^T$ gives

\[
\hat{P} [B -1_N] \begin{bmatrix}
r_s \\
||r_s||^2
\end{bmatrix} = \hat{P} b
\]

\[
= \hat{P} b + \hat{P} e
\]

and the solution for WOS is the following:

\[
\hat{\theta}_{WOS} = \left(B^T \hat{P}^T \hat{C}_e^{-1} \hat{P} B\right)^{-1} B^T \hat{P}^T \hat{C}_e^{-1} \hat{P} b.
\]

(44)

Note that the relationship between $\hat{C}_e$ [defined in (21) and $\hat{Q}$ defined in Section V] is found to be

\[
\hat{C}_e = \hat{P} \hat{Q} \hat{P}^T.
\]

(45)

Since the rank of $\hat{P}$ is $N-1$, $\hat{C}_e$ must be singular and the pseudoinverse of $\hat{C}_e$, $\hat{C}_e^T$, is used instead of $\hat{C}_e^{-1}$ as pointed out earlier in Section IV. By using SVD, $\hat{C}_e$ can be factored into

\[
\hat{C}_e = U S U^T.
\]

(46)

where $S$ is $N-1$ by $N-1$ diagonal matrix that contains nonzero eigenvalues of $\hat{C}_e$ and the columns of $U$ are corresponding eigenvectors. $\hat{P}$ and $\hat{Q}$ can be represented as partitioned matrices as follows:

\[
\hat{P} = \begin{bmatrix}
\hat{P}_0 & -\hat{P}_0 1_{N-1}
\end{bmatrix}
\]

(47)

\[
\hat{Q} = \begin{bmatrix}
Q_{11} & 0 \\
0 & Q_{22}
\end{bmatrix}
\]

(48)
where \( \tilde{Q}_{11} \) and \( \tilde{Q}_{22} \) are \((N-1) \times (N-1)\) and \(1 \times 1\) matrices, respectively, and \( \tilde{P}_0 \) is \(N(N-1)/2\) by \(N-1\) matrix with rank \(N-1\). Thus, \( \tilde{C}_e \) is given by

\[
\tilde{P}_0 \tilde{Q}_e \tilde{P}_0^T = \tilde{P}_0 \Sigma_0 \tilde{P}_0^T \tag{49}
\]

where \( \Sigma_0 \triangleq (\tilde{Q}_{11} + 1_{N-1} \tilde{Q}_{22} 1_{N-1}^T) \). Substituting (46) and (49) into (45) gives

\[
USU^T = \tilde{P}_0 \Sigma_0 \tilde{P}_0^T
\]

where \( U = (\tilde{P}_0^T U)^{-1} (U^T \tilde{P}_0)^{-1} \). Inserting (50) into (51) gives

\[
\tilde{C}_e = \tilde{C}_e^T \tag{50}
\]

The term \( \tilde{P}_0 \tilde{C}_e \tilde{P}_0^T \) in (44) can be written as a partitioned matrix as follows:

\[
\tilde{P}_0^T \tilde{C}_e \tilde{P}_0 = \begin{bmatrix}
\tilde{P}_0^T & 0
-1_{N-1} \tilde{P}_0^T & 1_{N-1} \tilde{P}_0^T
\end{bmatrix}
\begin{bmatrix}
\tilde{C}_0^T [\tilde{P}_0 & -\tilde{P}_0 1_{N-1}]
-1_{N-1} \tilde{C}_0^T & 1_{N-1} \tilde{C}_0^T
\end{bmatrix},
\tag{51}
\]

Inserting (50) into (51) gives

\[
\tilde{P}_0^T \tilde{C}_e \tilde{P}_0 = \begin{bmatrix}
\Sigma_0^{-1} & -\Sigma_0^{-1} 1_{N-1}
-1_{N-1} \Sigma_0^{-1} & 1_{N-1} \Sigma_0^{-1} 1_{N-1}
\end{bmatrix} \tag{52}
\]

\( \Sigma_0^{-1} \) can be expanded using the Woodbury’s identity as follows:

\[
\Sigma_0^{-1} = \tilde{Q}_{11}^{-1} - \frac{\tilde{Q}_{11}^{-1} \tilde{Q}_{22}^{-1} \tilde{Q}_{11}}{\tilde{Q}_{22}^{-1} + 1_{N-1} \tilde{Q}_{11}^{-1} 1_{N-1}} \tag{53}
\]

Substituting (53) and (52) in (51) gives

\[
\tilde{P}_0^T \tilde{C}_e \tilde{P}_0 = \tilde{Q}_0^{-1} - \frac{\tilde{Q}_0^{-1} \tilde{Q}_{22}^{-1} \tilde{Q}_0^{-1}}{\tilde{Q}_{22}^{-1} + 1_{N-1} \tilde{Q}_0^{-1} 1_{N-1}} \tag{54}
\]

Let us now consider the solution obtained from WD using the partitioned matrix form

\[
\hat{\theta}_{WD} = \left( A^T \tilde{Q}_e^{-1} A \right)^{-1} A^T \tilde{Q}_e^{-1} h \tag{55}
\]

\[
= \left( \begin{bmatrix} B^T & -1_N \end{bmatrix} \tilde{Q}_e^{-1} [B & -1_N] \right)^{-1} \left( \begin{bmatrix} B^T \\ -1_N \end{bmatrix} \right) \tilde{Q}_e^{-1} h. \tag{56}
\]

Considering only the estimate of \([r_s^T \ | \ [r_s]^2]^T\), we obtain

\[
\hat{\theta}_{WD[r_s^T \ | \ [r_s]^2]} = (B^T HB)^{-1} B^T HH \tag{57}
\]
where

\[
\frac{\partial \mu}{\partial \lambda_1} = \frac{\partial \mu}{\partial x_s} = \begin{bmatrix} 2\sigma^2_s(x_1 - x_s) + 2\sigma^2_s(x_N - x_s) \\ \|x_s - r_1\|^2 \\ \|x_s - r_N\|^2 \end{bmatrix}^T,
\]

\[
\frac{\partial \mu}{\partial \lambda_2} = \frac{\partial \mu}{\partial y_s} = \begin{bmatrix} 2\sigma^2_s(y_1 - y_s) + 2\sigma^2_s(y_N - y_s) \\ \|y_s - r_1\|^2 \\ \|y_s - r_N\|^2 \end{bmatrix}^T,
\]

\[
\frac{\partial \mu}{\partial \lambda_3} = \frac{\partial \sigma^2_s}{\partial \sigma^2_s} = \begin{bmatrix} 1 \\ \|x_s - r_1\|^2 \\ \|x_s - r_N\|^2 \end{bmatrix}^T.
\]

Substituting (64) into (63), F can be obtained and the CRB is given by $F^{-1}$.

**APPENDIX IV**

This Appendix shows that when the source is white, WDC attains the CRB. The notations are the same as in Appendix IV unless defined. When the source is assumed to be white, $\nu = 0$, the unknown parameters is now defined as $\hat{\Theta} = [x_s, y_s, \sigma^2_s]^T$, and the covariance matrix, $K$, becomes $2/L\text{diag}\{\mu^2_1, \ldots, \mu^2_N\}$.

$$ K = \frac{2}{L} \text{diag} \{ \mu_1^2, \ldots, \mu_N^2 \} $$

$$ \frac{\partial K}{\partial \hat{\Theta}} = \text{diag} \{ \frac{4\mu}{L} \} \frac{\partial \hat{\Theta}}{\partial \hat{\Theta}} \cdot \text{where the symbol } \odot \text{ represents Schur (element-by-element) product.} $$

The CRB is thus, given by

$$ F^{-1} = \left( 1 + \frac{4}{L} \right) (H^T K^{-1} H)^{-1} \tag{65} $$

where

$$ H = \begin{bmatrix} 2\sigma^2_s (x_s - r_1)^T \\ \|x_s - r_1\|^2 \\ \|x_s - r_N\|^2 \end{bmatrix} \cdot \frac{1}{\|x_s - r_N\|^2} $$

To derive the covariance matrix of the WDC solution, we first consider the covariance matrix of the estimation $\hat{\Theta}^2$ in (33) which is

$$ \Sigma = (J^T J)^{-1} \tag{67} $$

Let the elements of $\hat{\Theta}_{WDC}$ be in the form $x_s + e_x, y_s + e_y$, and $\sigma^2_s + e_{\sigma}$. The estimates of the elements of $\hat{\Theta}^2$ in (33) can be given by

$$ \hat{x}_s^2 = (x_s + e_x)^2 \approx x_s^2 + 2x_se_x \tag{68} $$

$$ \hat{y}_s^2 = (y_s + e_y)^2 \approx y_s^2 + 2y_se_y \tag{69} $$

$$ \hat{\sigma}_s^2 = \sigma^2_s + e_{\sigma} \tag{70} $$

The approximation is valid when $e_x$ and $e_y$ are small. Therefore, $[e_x, e_y, e_{\sigma}]^T = \hat{B}^{-1} \Delta \Theta^2$ where $\Delta \Theta^2 = \Theta^2 - \hat{\Theta}^2$ and $\hat{B} = (\text{diag}\{2x_s, 2y_s, 1\})$. The covariance matrix of $\hat{\Theta}_{WDC}$ is, thus, found to be $C_{WDC} = \hat{B}^{-1} \Sigma \hat{B}^{-1}$ where $\Sigma$ is defined in (67). Substituting $\Sigma$ yields

$$ C_{WDC} = (\hat{B}J^T \hat{B}^{-1} A A^T C_{-1} \hat{B}^{-1} A B^{-1} J B)^{-1}. \tag{71} $$

In the white source case, $C_{W}$ becomes $\text{DKD}$ where $D = \text{diag}(\|r_s - r_1\|^2, \ldots, \|r_s - r_N\|^2)$ (see Section V). Substituting $C_{W}$ in (71) yields

$$ C_{WDC} = (\hat{B}J^T \hat{B}^{-1} A A^T \hat{B}^{-1} A B^{-1} J B)^{-1}. \tag{72} $$

Multiplying the matrices gives $D^{-1} A B^{-1} J B = -H$. Therefore, $C_{WDC} = (H^T K^{-1} H)^{-1} \approx F^{-1}$ when $L$ is assumed to be sufficiently large.

**REFERENCES**


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