Dynamical systems
Dynamical Systems Analysis

- Isaac Newton and Gottfried Leibniz, in the late 17th, developed an outlook, still at the basis of our science and technology, on how to understand our world.
- Task Dynamics is the application of their ideas, and the ideas implicit in 300 years of development of their conceptual framework, to how we speak.
- Here’s their conceptualization of the world we live in:
  1. There are **Phenomena of Nature**: Position of planets\(t\) and articulators \(t,x\), voltage of neurons\(t,x\), acoustic waves \(t\), unemployment time series \(t,x\), stock market time series\(t\), the amount of plastic in the ocean\(t,x\), test scores \(SES,district\) etc. **Phenomena of Nature are functions**: *Dependent Variable functions of independent variables*.
  2. There are **Laws of Nature**: simple theoretical entities that predict the phenomena around us. *Mathematically, they are differential equations (also called dynamical systems)*.
  3. **Solution**: We can predict Phenomena of Nature functions by *solving* Law of Nature Differential Equations for functions in the world, which can be related to measurable functions.
Example Natural Phenomenon function: digestion of food

Undigested food, Digested food, and Enzyme

Notice: both functions are variable as a function of time.

https://en.wikipedia.org/wiki/Michaelis\-Menten_kinetics
Natural Law: Slope of DF is related to Slope of UF

- Simplified *invariant* natural law of digestion: Digested food increases as Undigested food decreases.
- Despite functional variation there is an invariant relation relating the *slopes* of the functions with time: As one increases, the other decreases, regardless of the point in time, how much you ate, the last time you ate. etc. That's what it means for natural laws to be simple: it is invariant.
Michaelis-Menten Laws

- Note what the natural law is about: **relation of slopes**. The law is not about the instantaneous values of the function, but about the relation of their slopes.
- This is a general feature of natural laws according to the Newtonian/Leibnizian notion of natural law, which will relate the dependent variables, their slopes, and curvatures lawfully.
- This specific law, perhaps the most fundamental in biochemistry, was developed by Leonor Michaelis and his student Maud Menten.
- It invariantly describes chemical reactions in big oceans, reactors, kitchens, and cells.
Example: Hourglass dynamical system

As time progresses, unit by unit, the same amount of sand falls in every unit of time. The amount of the sand that drops depends on the properties of the sand and the size of the hole. We can make fast and slow hourglasses.
Hourglass dynamical system: Slope $= \frac{x_{\text{end}} - x_{\text{begin}}}{t_{\text{end}} - t_{\text{begin}}} = -k$

$\text{Slope} = \frac{0 - 10}{6 - 1} = -2$

$\text{Slope} = \frac{0 - 10}{4 - 1} = -3.3$

$\text{Slope} = \frac{0 - 10}{2 - 1} = -10$
Hourglass dynamical system: \( \text{Slope} = \frac{x_{\text{end}} - x_{\text{begin}}}{t_{\text{end}} - t_{\text{begin}}} = -k \)

- Again: Note what is invariant, not \( x \), but the slope of \( x \).
- Leibniz denoted the slope at a point of a function \( x(t) \) as \( \frac{dx}{dt} \), with \( d \) standing for the \textit{différance} in the numerator and denominator of \( \frac{x_{\text{end}} - x_{\text{begin}}}{t_{\text{end}} - t_{\text{begin}}} \).
- Therefore we can write the differential equation of the hourglass as \( \frac{dx}{dt} = -k \), with larger \( k \) denoting a bigger hole in the hourglass.
Example Falling Object dynamical system: $\frac{dx}{dt} = -9.8t$

Galileo and Descartes: When you drop an object, it travels downwards faster and faster as time progresses.

$t = 0s, v = 0m/s$  
$t = 1s, v = 9.8m/s$  
$t = 2s, v = 19.6m/s$  
$t = 3s, v = 29.4m/s$  
$t = 4s, v = 39.2m/s$
Example Falling Object dynamical system: \[ \frac{dx}{dt} = -9.8t \]

The function is an upside down parabola!
Viscoelastic tissue law: $\frac{dx}{dt} = -0.33x$

If you pull on a tissue, it goes back to its original position, first very fast, then slower and slower. The velocity of the tissue is related through a negative coefficient to the position. If the position is very far from the neutral point in the positive direction, the velocity is negative, whereas if the position is negative, the velocity is positive. Despite the fact that what happens at each point is different, the law is fixed.

These functions have a name: Negative Exponential.
So far we’ve covered 2 of the 3 aspects of the Newton-Leibniz ontology: Functions and invariant relations relating dependent variable, its slope, its curvature, and the independent variable through an invariant differential equation.

- **Hourglass**: \( \frac{dx}{dt} = k \)
- **Gravity**: \( \frac{dx}{dt} = -kt \)
- **Spring**: \( \frac{dx}{dt} = -kx \)
Linear dynamical systems

- All of these systems are linear because the slope of relation between state (x or y or t) and change in state (dx, dy) is a linear function with a fixed slope.

Hourglass

\[ \frac{dx}{dt} = k \]

Gravity

\[ \frac{dx}{dt} = -9.8t \]

Spring

\[ \frac{dx}{dt} = -kx \]
Dynamical Constraints  (Gafos & Benus, 2006)

- Dynamical systems that model the preferences of the grammar to select certain phonetic states (e.g. $x_0$)

- A linear first-order system exhibits an attractor along some continuous state variable $x$, that represents a task variable, e.g. Lip Aperture.

- All trajectories converge at LA = -2.

- IDEA: The grammar produces only one stable value for this unit, no matter what kind of “input” the grammar begins with.

- ALTERNATE IDEA: Planning process derives only one goal for the unit, regardless of how planning is initiated.

$$\frac{dx}{dt} = -k(x - x_0)$$

\[ x_0 = -2 \]

\[ k = .95 \]
• Attractors of a dynamical system of this type can be visualized by plotting its potential function $V(x)$.

$$V(x) = \frac{k(x - x_0)^2}{2}$$

• The potential is obtained by integrating the differential equation.

$$\frac{dx}{dt} = -k(x - x_0)$$

• So therefore, the change of $x$ in time is equal to the negative slope of the potential function.

$$\frac{dx}{dt} = -\frac{d(V(x))}{dx}$$

• Or, another law of slopes: The slope of $x$ with respect to the independent variable ($t$) is equal to the slope of the potential function at the value of $x$.

- Imagine dropping a ball on the potential function landscape.
- Where the slope is non-zero, there will be change in state towards the attractor.
- At the attractor, the slope is zero and there will be no more change.
% first_order.m

clear x

dt = .001;
k = .95;

niter = 4000;

x(1) = 3;
dx(1) = 0;
x0 = -2;

for i = 2:niter
    dx(i) = -k.*(x(i-1)-x0)*dt;
x(i) = x(i-1)+dx(i);
end

subplot (1,2,1), plot ([2:niter]*dt, x(2:end), 'r', 'LineWidth', 3);
ylabel ('x', 'FontSize',18);
xlabel ('t', 'FontSize',18);

x=[-3:.01:3];
dx = -k.*(x-x0)*dt;

subplot (1,2,2), plot (x, dx, 'g', 'LineWidth', 3);
ylabel ('dx', 'FontSize',18);
xlabel ('x', 'FontSize',18);
hold on
plot(x,zeros(1,length(x)), 'k', 'Linewidth', 2);
xlim([-3 3]);
hold off

Attractor:
Stable equilibrium
Negative slope