1. Digital Resonators

Digital resonators are second-order recursive filters. Second order means that the number of coefficients (L in eq. 8 of the Recursive Filter handout) is two. This means that the a(2) coefficient—the weight of output sample Y(k-2)—is non-zero. Usually, the a(1) coefficient is also non-zero. The amplitude response of such filters are characterized by single peaks (the resonance frequency) of variable width (bandwidth). The impulse response is a damped sinusoid. Within the source-filter model of speech, each formant of the vocal tract filter can be treated as a digital resonator with a particular frequency and bandwidth. Thus, each formant can be specified with two filter coefficients. We will derive the relation between the coefficients and the frequency and bandwidth of the resonator (and vice versa).

2. An example of a digital resonator:

(1) \( Y(k) = X(k) - a(1) Y(k-1) - a(2) Y(k-2) \)

As a particular example, choose \( a(1) = -1.287 \) and \( a(2) = .8282 \)

To implement this filter within MATLAB, set:
\[
\begin{align*}
    b &= 1 \\
    a &= [1 \hspace{0.5cm} -1.287 \hspace{0.5cm} .8282]
\end{align*}
\]

Using these values to filter an impulse results in the following output:
Note that the impulse response is a damped sinusoid, i.e., a resonator.

The amplitude response of this filter can be calculated (using the function `transfer(b, a)`), and it looks like this:

![Amplitude response graph](image.png)

What is the resonance frequency of this digital resonator in radians/sample. If the sampling rate were 10K, what would the resonance frequency be in Hz?

In general, the transfer function of a digital resonator can be obtained from the expression in (2). Note that this is just a special case of equation (9) in the Recursive Filter handout, in the special case where M=1 and L=2.

\[
(2) \quad H(z) = \frac{b(1)}{1 + a(1)z^{-1} + a(2)z^{-2}}
\]

The frequency and bandwidth characteristics of a digital resonator are determined by the values of a(1) and a(2), as we now show.

3. Poles

3.1 Roots of the H(z) denominator

Consider the equation in (2). There are certain values of z for which the expression in the denominator of (2) will be zero (the roots of the polynomial). They are found by setting the expression in the denominator to 0 and solving for z. We will see in the next section how to actually solve the expression for z. For the moment, however, just imagine
that we already have done so and know the values of these roots. Expression (2) can be rewritten in terms of the roots as (3):

\[ H(z) = \frac{b(1)}{(z - \text{root}(1)) \cdot (z - \text{root}(2))} \]

Note that if \( z \) is equal to either of the roots, then the value of the denominator is 0. Also, the value of \( H(z) \) goes to infinity as the value of the denominator goes to 0. Thus, the roots of the denominator will determine the values of \( z \) for which \( H(z) \) becomes infinite (or very large). Since the roots will, in general, be complex, we can rewrite (3) as (4a). Since the roots also occur in complex conjugate pairs (as we will see below) this expression can be written as(4b).

\[ H(z) = \frac{b(1)}{(z - R_1 e^{j\theta_1}) \cdot (z - R_2 e^{j\theta_2})} \]

Thus, both roots have a magnitude of \( R \). The argument of one is \( \theta \), while the other is \(-\theta\).

3.2 Resonator frequency
What can we learn about the frequency response of a resonator from the location of the poles in the complex plane? The figure below shows the location of the poles for the example discussed in section 2 above. The poles are marked with an ‘x’ within a little circle.

Once again, we can interpret frequencies as points on the unit circle (points for which \( z = e^{j\omega} \)). In general, the magnitude of the \( H(z) \) for any frequency will be inversely proportional to its distance from the poles (actually the product of its distances from all the poles). We will not derive this formally here, but it should appear reasonable. In the present example (with just
one complex conjugate pair of poles), we can ignore the negative frequencies (and the pole that lies below the real axis). The positive frequency with the smallest distance from the pole will be one for which \( \omega = \theta \). Thus, the resonant frequency of a digital resonator (\( \omega \)) is equal to the argument of the roots of the polynomial.

Note that in this case, \( \theta \) appears to be about \( \pi/4 \) radians (45 degrees). Look again at the amplitude response of the filter and you will see that it has a peak at a frequency of \( \pi/4 \) radians/sample.

### 3.3 Resonator Bandwidth

The magnitude of the roots, \( R \), is related to the bandwidth of the resonator. This relation can be visualized in the figure below.

The two examples (left and right) differ only in the magnitude of the roots--\( R \) on the left and \( r \) and the right. The argument of the roots (\( \theta \)), and thus the resonant frequency, is the same in both cases. Now compare the magnitude of \( H(z) \) for the frequency points marked with black dots and those marked with grey dots, corresponding to the resonant frequency, \( \omega \), (black dots) and frequencies that are somewhat smaller and greater than \( \omega \). In the example on the left, where \( R \) close to 1, there will be a large difference in the magnitude of \( H(z) \) between the black dot and the grey dots. The black dot is almost on top of the ‘x’, while the grey dots lie at a considerable distance from it. Thus, the magnitude of the \( H(z) \) will drop off considerably for frequencies on either side of the resonance frequency (in this example, about \( \pi/8 \) on either side of the resonant frequency). This would constitute a fairly narrow bandwidth. In the example on the right, however, the distance from the black dot to the ‘x’ is fairly similar to the distance from the grey dots to the the ‘x.’ Thus, in this case there will be less drop off in magnitude of \( H(z) \) for frequencies on either side of the resonance. This case will correspond to a wider bandwidth than the case on the left. Thus, as the magnitude of the root approaches 1, the bandwidth decreases.

Mathematically, the relation between \( bw \) (bandwidth in radians/sample) and \( R \) (the magnitude of the roots) is given in (5):
(5) \[ R^2 = e^{-bw} \quad \text{or} \quad bw = -2 \cdot \ln (R) \]

4. Calculation of frequency and bandwidth of a digital resonator

We can calculate the frequency and bandwidth of a digital resonator if we know the value of \( \theta \) and \( R \) for the roots. How can we calculate the roots?

A quadratic equation (a second order polynomial), given in (6), has the roots given in (7) (the quadratic formula):

(6) \[ ax^2 + bx + c = 0 \]

(7) \[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

The denominator in (2) looks similar to (6), but with negative powers of \( z \) instead of positive powers. We can remedy this by multiplying the numerator and denominator of (2) by \( z^2 \), giving (8):

(8) \[ H(z) = \frac{b(1)z^2}{z^2 + a(1)z + a(2)} \]

Thus, we can use the quadratic formula to solve for the roots of the denominator of (8), where \( a=1 \), \( b=a(1) \) and \( c=a(2) \).

There are two cases to consider. When \( b^2 - 4ac \) is greater than or equal to 0, then the roots are real. This is the uninteresting case: the resonant frequencies would be either 0 or \( \pi \). The more interesting case is where \( b^2 - 4ac \) is less than zero. Then the roots are complex, and the real and imaginary parts of the roots are given in (9):

(9) \[
\begin{align*}
\text{Real part} & = \frac{-b}{2a} \\
\text{Imag part} & = \pm \frac{\sqrt{-b^2 + 4ac}}{2a}
\end{align*}
\]

(Note that the terms under the radical are the negative of those in (7). That’s because the imaginary part is multiplied by \( j = \sqrt{-1} \)).

Now substituting the coefficients in (8) for \( a \), \( b \), and \( c \) we get:

(10) \[
\begin{align*}
\text{Real part} & = \frac{-a(1)}{2} \\
\text{Imag part} & = \pm \frac{\sqrt{-a(1)^2 + 4a(2)}}{2}
\end{align*}
\]
Now we know that the magnitude squared ($R^2$) of a complex number is equal to the sum of the real part squared plus the imaginary part squared. Thus, we can calculate $R$:

\[
R^2 = \left( \frac{-a(1)}{2} \right)^2 + \left( \frac{\sqrt{-a(1)^2 + 4a(2)}}{2} \right)^2
\]

\[
R^2 = \frac{a(1)^2}{4} + \frac{-a(1)^2 + 4a(2)}{4}
\]

\[
R^2 = \frac{4a(2)}{4}
\]

\[
R = \sqrt{a(2)}
\]

If we know $R$ and the Real part, we can also then derive $\theta$, as shown in (12):

\[
\cos \theta = \frac{\text{Real part}}{R}
\]

\[
\cos \theta = \frac{-a(1)}{2R}
\]

\[
\cos \theta = \frac{-a(1)}{2\sqrt{a(2)}}
\]

\[
\theta = \arccos \left( \frac{-a(1)}{2\sqrt{a(2)}} \right)
\]

Thus, given the coefficients of a digital resonator, we can calculate the magnitude ($R$) and argument ($\theta$) of the roots. From these, we can calculate the bandwidth ($bw$) of the resonator, as in (13). Its frequency ($\omega$) is just equal to $\theta$.

\[
R^2 = e^{-bw}
\]

\[
a(2) = e^{-bw}
\]

\[
bw = -\ln(a(2))
\]